

Whitehead's Category of the Ultimate as Lawvere's Topos in Category Theory

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With thanks to late colleague Michael Heather

Process and Reality

- *An Essay in Cosmology*
 - subtitle given by Alfred North Whitehead to his celebrated Gifford lectures:
 - *PROCESS & REALITY (PR)* at Edinburgh in the session of 1927-28.
 - Published as Whitehead, Alfred North, *Process and Reality: An Essay in Cosmology*, corrected edition, edd D R Griffin, D W. Sherburne, New York, The Free Press (1978).
 - <https://archive.org/details/AlfredNorthWhiteheadProcessAndReality>
 - Two conflicting original editions, poor proof reading, inconsistencies, resolved as far as possible in corrected edition.

Process and Reality 2

- His cosmology is developed in terms of a Categorical Scheme
 - Philosophy of Organism.
 - Described as his speculative philosophy.
 - The foundation of his whole scheme of cosmology is the *Category of the Ultimate*.
 - A category in process terms is a typing and this fundamental category of his “expresses the general principle presupposed in the three more special categories”

Earlier Work

- Alfred North Whitehead's Process and Reality
 - Alex Scott
 - <https://www.angelfire.com/md2/timewarp/whitehead.html>
- Robert Rosen's Life Itself
 - Explores categorical relations in biological systems
- Heather & Rossiter:
 - Add ANPA 2023 paper
 - Heather, M, & Rossiter, N, The Topos of Category Theory and Reality, Proc XVII International Conference on the Science and Quality of Life, June 29 - July 2 2013, Vilnius, Lithuania, edited Romuald Brazis, Studium Vilnense A 11 204-213 (2014).
 - Heather, Michael, & Rossiter, Nick, The Process Category of Reality, ANPA 31, Cambridge 224-262 (2011).
 - Heather, Michael, & Rossiter, Nick, Adjoint Typing: Is it Whitehead's Category of the Ultimate? 7th International Whitehead Conference January 5-9, Bangalore, Karnataka, India pp.72-74 (2009).
 - Heather, Michael, & Rossiter, Nick, Process Category Theory, Salzburg International Whitehead Conference, University of Salzburg 3-6 July (2006).

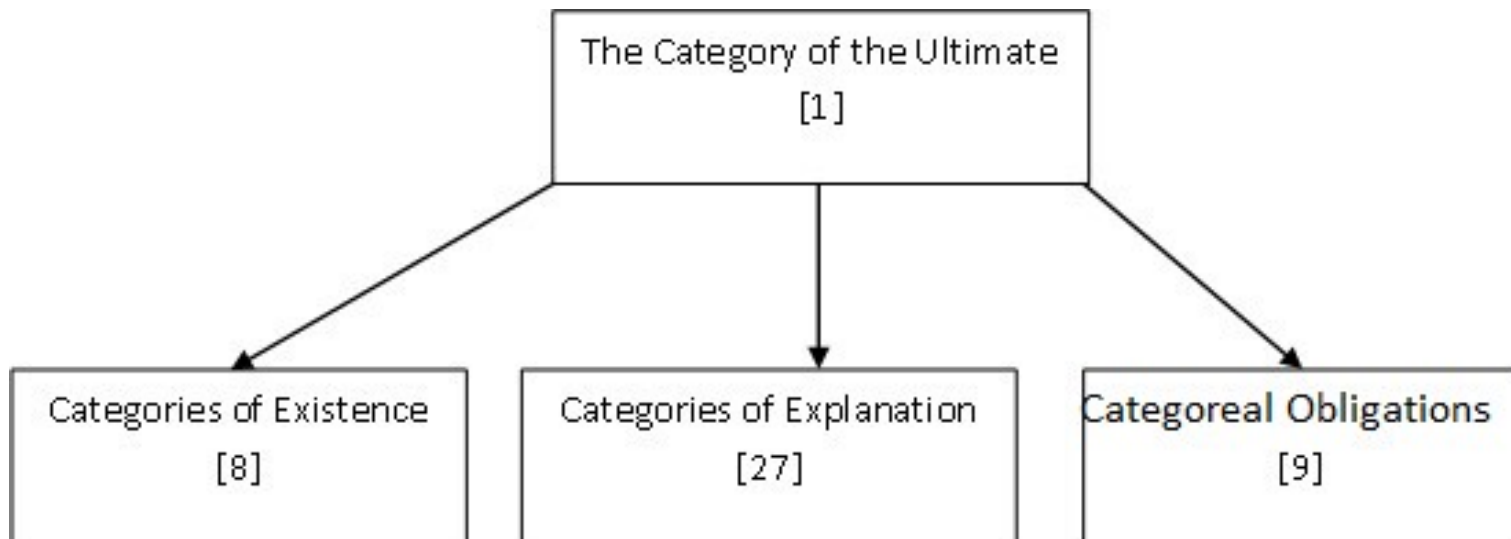
Rationale

- Whitehead's writing style is dense, involving many of his own terms
- Closer look is warranted on his actual text:
 - Increased understanding of category theory
 - Focus on concepts ANW considered important
- Desire to extend formalisms into biological sciences
 - ANW unifies the physical and biological

The Categoreal Scheme

- The special categories are:
 - Category of the Ultimate
 - Categories of Existence
 - Categories of Explanation
 - Categoreal Obligations
- These special categories are composed of
 - eight categories of existence
 - twenty seven categories of explanation
 - nine categoreal obligations
- “The Category of the Ultimate expresses the general principle presupposed in the three more special categories” p.21
- The whole of PR rests on this categoreal scheme

At first sight there seems to be a hierarchical typing relationship among these categories which might look like this:



Numbers are count of categories of each type

What Whitehead does not say

- However, Whitehead does not provide such a diagram in PR.
- Nor does he state that there is a hierarchical relationship.
- Whitehead does not even explain what he means by the term 'category'.
 - It seems it is defined by the Category of the Ultimate itself and therefore is self-referencing.
 - May be Aristotelian p.30
 - “ This Category of the Ultimate replaces Aristotle's category of 'primary substance’ “ p.21 (classify, predicate, extension)
- If he has no difficulty with a category being a member of itself,
 - then a category is not to be identified with a set,
 - the concept earlier promoted in his other *magnum opus* (co-authored with Bertrand Russell) the *PRINCIPIA MATHEMATICA (PM)*.

Speculative Nature

- Whitehead seems very conscious of the speculative nature of his philosophy at this stage.
- The whole of Part I of *PR* is headed 'a Speculative Scheme'.
- It is speculative perhaps because at the time
 - he was giving the Gifford lectures and
 - for the remaining 20 years of his life
 - there was no formal presentation available for *PR*
 - as he and Russell were able to provide for *PM*.

One-Substance Approach

- Descartes (and other philosophers) considered there were two substances
 - Material body
 - Mental (mind, thought)
- Whitehead abandoned this approach, considering there was just one substance
 - So unifying mind and matter as a single process
- Whitehead describes his system of speculative philosophy as a ‘philosophy of organism’ or ‘organic philosophy,’ since he views reality as consisting of interrelated and mutually dependent parts that are involved in sustaining vital processes

Perhaps no longer a speculative philosophy

- New formal techniques are available
- So this speculative state of affairs may no longer hold.
- There is now a formal theory of categories only just beginning at the time of Whitehead's death in 1947 but now maturing

CT Foreshadowed by ANW

- Category Theory is a theory foreshadowed in Whitehead's *Category of the Ultimate*
 - quite comprehensively in the sense of his preface to *PR* at p. vi:
- Motivation for a complete cosmology
 - to construct a system of ideas which bring the aesthetic, moral and religious interest into relation with those concepts of the world which have their origin in natural science.
 - [Whitehead *PR* Part I]

Problems with Hierarchy

- From the formal theory of categories
 - Can understand the need for interdependence between categories
 - Not achieved in a hierarchy
 - So Whitehead presumably dismissed the use of hierarchies.
- The relationship is more complex than the hierarchy
 - in the same way Russell used the phrase ‘ramified type-theory’ rather than ‘hierarchical type-theory’ although both words contain the sense of a tree.

Eight Categories of Existence

Whitehead [PR p.22] as:

- (i) Actual Entities (also termed Actual Occasions), or Final Realities, or Res Verae
- (ii) Prehensions, or Concrete Facts of Relatedness
- (iii) Nexūs (plural of Nexus), or Public Matters of Fact
- (iv) Subjective Forms, or Private Matters of Fact
- (v) Eternal Objects
- (vi) Propositions, or Matters of Fact in Potential Determination, or Impure Potentials for the Specific Determination of Matters of Fact, or Theories.
- (vii) Multiplicities, or Pure Disjunctions of Diverse Entities
- (viii) Contrasts, or Modes of Synthesis of Entities in one Prehension, or Patterned Entities

Subjective Forms: Entities, Prehension and Concrecence

Entities

- Real or Actual (exist)
- Individual (atomic)
- Particular (singled out, identity)
- Can be joined together as a nexus (union of similar entities, ordered society)
- “Every entity should be a specific instance of one category of existence” PR p.20 (classification)
- Can be seized by prehension (product or coproduct relatedness) to give a subjective form

Prehension

- An overloaded word
 - Grasping, seizing
 - An interaction of a subject with an event or entity which involves perception but not necessarily cognition.
- “There are eight Categories of Existence (PR p.22):
 - (i) Actual Entities (also termed Actual Occasions), or Final Realities, or Res Verae [true thing].
 - (ii) Prehensions, or Concrete Facts of Relatedness.”

Prehension – Conceptual Definition

PR p.19

“A prehension reproduces in itself the general characteristics of an actual entity: it is referent to an external world, and in this sense will be said to have a 'vector character'; it involves emotion, and purpose, and valuation, and causation. In fact, any characteristic of an actual entity is reproduced [29] in a prehension. It might have been a complete actuality; but, by reason of a certain incomplete partiality, a prehension is only a subordinate element in an actual entity.”

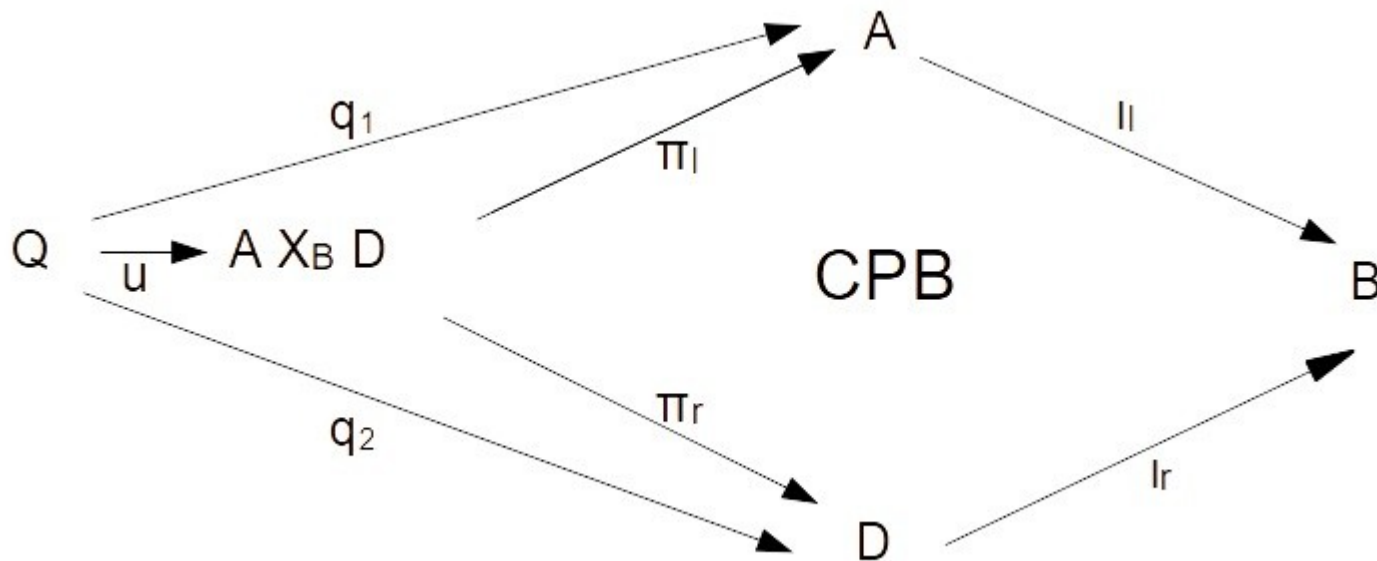
Prehension – More Concrete

Definition PR p.23

“(xi) That every prehension consists of three factors: (a) the 'subject' which is prehending, namely, the actual entity in which that prehension is a concrete element; (b) the 'datum' which is prehended; (c) the 'subjective form' which is how that subject prehends that datum.”

- This is a data relationship, either X or $+$.
- A pullback (or pushout) category:
 - (c) is (a) $X_{(b)}D$ (D is a third entity)
 - (c) is the subjective form
- Cartesian closed (X) (or Cocartesian closed ($+$)) category

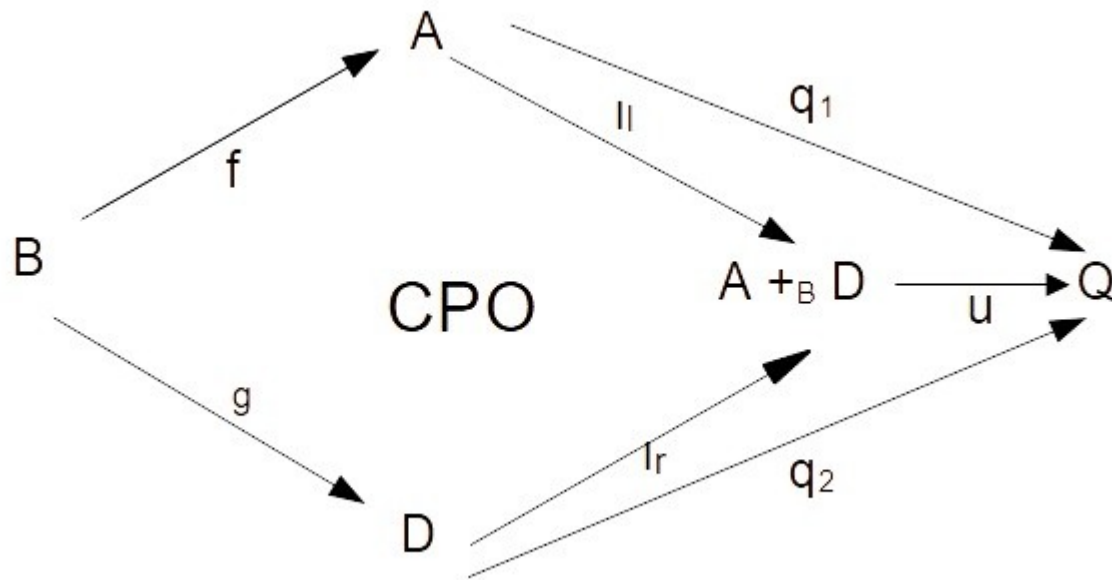
Category CPB: the Limit Diagram of a Category C: the Pullback $A \times_B D$



The subjective form of Whitehead is the
product $A \times_B D$

$A = (a)$, $B = (b)$, $(c) = A \times_B D$, D is introduced
If B is an Actual Entity, prehension is physical

Category CPO: the Colimit Diagram of a Category C:
the Pushout $A +_B D$

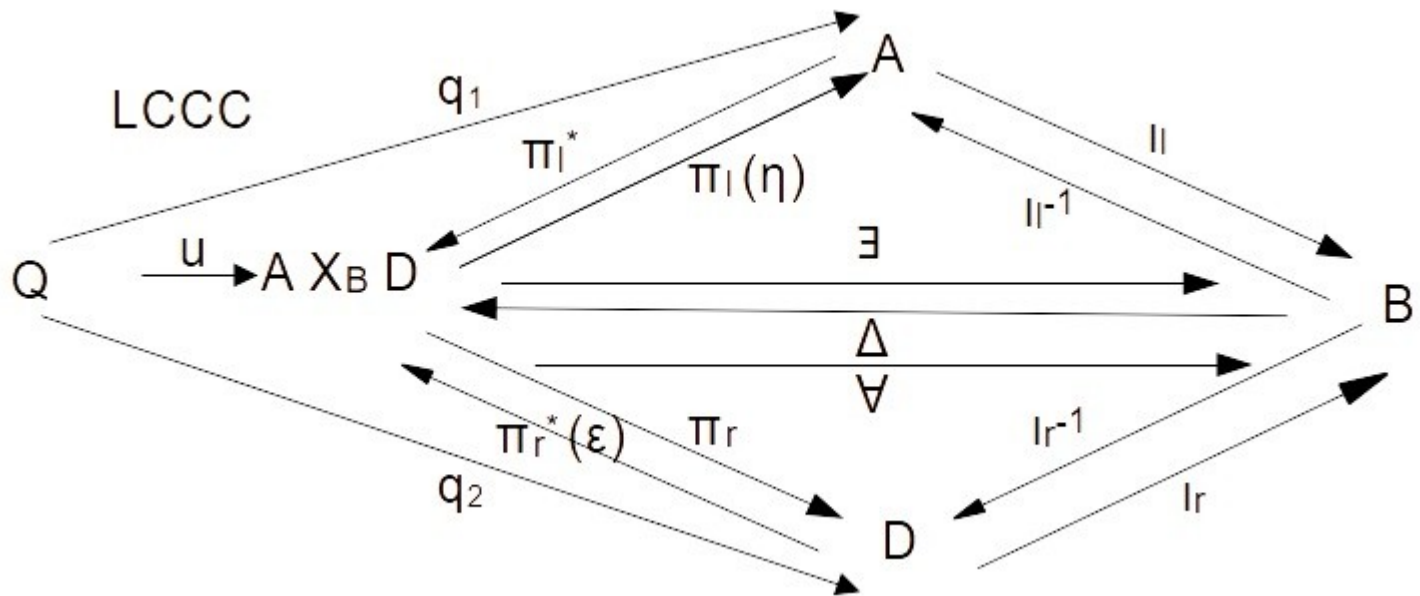


Concrescence

- Prehension is a process by which an actual entity, or prehending subject, becomes itself by appropriating elements from other actual entities.
- The becoming of an actual entity occurs through a concrescence of prehensions.
- Satisfaction is a final phase of concrescence (or the process of integration of feeling), in which prehensions are integrated into a concrete unity.

Physical Prehension with Concrecence

The locally Cartesian closed category LCCC:
 Adjointness in the functors between the
 product $A \times_B D$ and B



The hyperdoctrine of Lawvere (1969): $\exists \dashv \Delta \dashv \forall$
 Adjointness

Adjointness gives Viability

Whitehead:

- Prehension gives relatedness (being)
- Concrecence gives viability of relations (becoming)

Categories:

- Pullbacks give relations
- Adjointness gives viability

Whitehead and Adjointness

- Whitehead later provides further information which indicates that he had the modern concept of adjointness in mind:
- There is a prevalent misconception that ‘becoming’ involves the notion of a unique seriality for its advance into novelty. This is the classic notion of ‘time’ which philosophy took over from common sense. [PR p.35]
- The ‘prehension’ of one actual entity by another actual entity is the complete transaction, analysable into the objectification of the former entity as one of the data for the latter, and into the fully clothed feeling whereby the datum is absorbed into the subjective satisfaction – ‘clothed’ with the various elements of its ‘subjective’ form. [PR p.52].

Adjointness is Becoming

- The first part is consistent with our understanding of LCCC: adjointness is not serial, but is a snap.
- The second part shows how adjointness provides satisfaction through Lawvere's hyperdoctrine as the final stage in becoming.

A Nexus of Entities with Defining Characteristic

Nexus is a Union

- Togetherness among actual entities is called a `nexus' [PR p.20 (classification)]
- A nexus appears to be a union of connected entities such as an ordered society.

Nexus - Expansion

- Whitehead gives further information on nexus at [PR p.24]:
- (xiv) That a nexus is a set of actual entities in the unity of the relatedness constituted by their prehensions of each other, or — what is the same thing conversely expressed - constituted by their objectifications in each other.
- (xv) That a proposition is the unity of certain actual entities in their potentiality for forming a nexus, with its potential relatedness partially defined by certain eternal objects which have the unity of one complex eternal object. The actual entities involved are termed the 'logical subjects', the complex eternal object is the 'predicate'.

Nexus – Order

- Whitehead provides more detail at [PR p.34], likening a nexus to a society with order:
- The notions of 'social order' and of 'personal order' cannot be omitted from this preliminary sketch. A 'society' in the sense in which that term is here used, is a nexus with social order; and an 'enduring object' or 'enduring creature' is a society whose social order has taken the special form of 'personal order.'

Nexus – Defining Characteristic

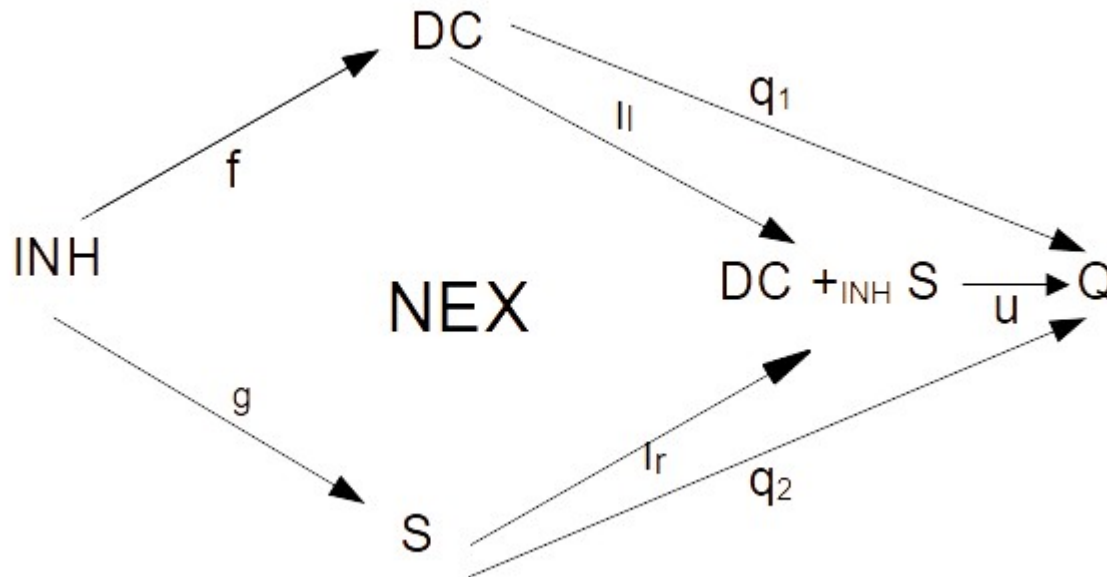
- A nexus enjoys 'social order' where (i) there is a common element of form illustrated in the definiteness of each of its included actual entities, and (ii) this common element of form arises in each member of the nexus by reason of the conditions imposed upon it by its prehensions of some other members of the nexus, and (iii) these prehensions impose that condition of reproduction by reason of their inclusion of positive feelings of that common form.
- Such a nexus is called a 'society' and the common form is the 'defining characteristic' of the society. The notion of 'defining characteristic' is allied to the Aristotelian notion of 'substantial form'.
- Thus the nexus forms a single line of inheritance of its defining characteristic.

Nexus as a Pushout Category

- Since sets are not inherently ordered, Whitehead is clearly not looking for a set-based solution. Categories have an inherent ordering, more naturally suited for representing a social order.
- The common form for a society, its defining characteristic, is likened to Aristotle's substantial form. Such a form is an Aristotelian second substance, in our terms the intension or class of a collection of values.
- So a nexus NEX is a pushout category $DC +_{INH} S$ whose contents are defined by an intension or type-category with defining characteristic DC, a society category S and a linking inheritance category INH creating the directed sum for pairs of DC and S.
- It is apparent that we are dealing with union here rather than product

Nexus as a Colimit

The Colimit Diagram of Category NEX: the Pushout $DC +_{INH} S$



S = Society, DC = Defining Characteristic, INH = line of inheritance

Multiplicities

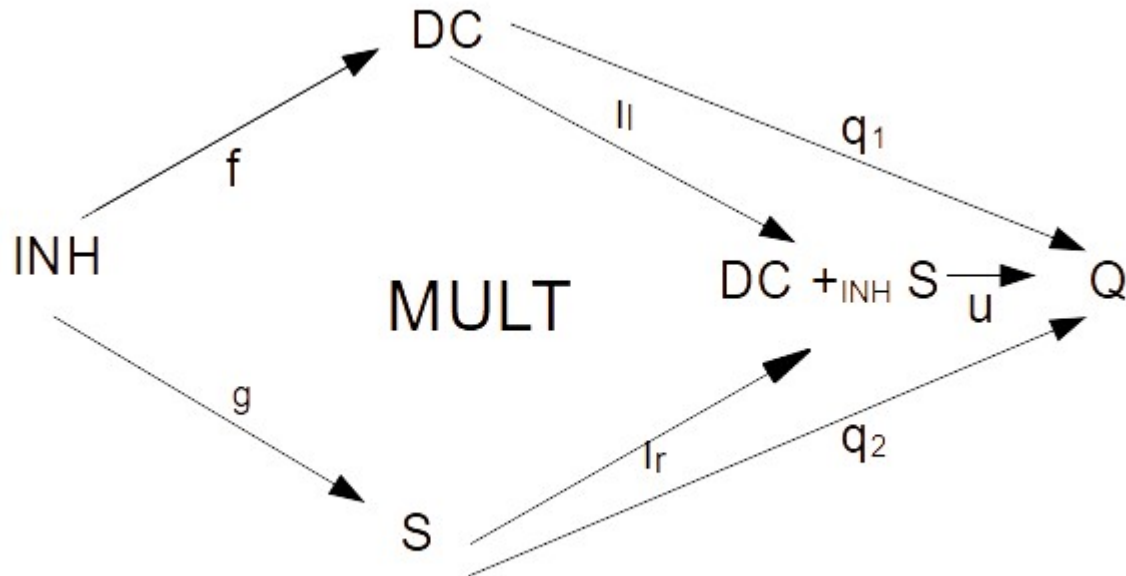
Multiplicities as Heterogeneous Collections

- Multiplicities, or Pure Disjunctions of Diverse Entities. Further explanation of this term is presented at [P&R p.24]:
- (xvi) That a multiplicity consists of many entities, and its unity is constituted by the fact that all its constituent entities severally satisfy at least one condition which no other entity satisfies. Every statement about a particular multiplicity can be expressed as a statement referent either (a) to all its members severally, or (b) to an indefinite *some of* its members severally, or (c) as a denial of one of these statements.

Multiple Inheritance

- The phrase 'at least one condition' indicates that multiple conditions occur but that at the lower extreme of one, the condition becomes closer to the defining characteristic of a nexus.
- Multiplicities are similar to nexus but in the latter we have a single defining characteristic for providing a type to the members of a society. A multiplicity therefore contains heterogeneous entities with no defining type characteristic. Whitehead uses the term single inheritance for the defining characteristic, and multiple inheritance for a multiplicity.

The Colimit Diagram of Category MULT: the Pushout $DC +_{INH} S$



Similar to NEXUS but g is 1:1 in NEX, N:M in MULT

Eternal Objects

Eternal Objects and Becoming

- (iii) In becoming of an actual entity, novel prehensions, nexus, subjective forms, propositions, multiplicities, and contrasts, also become; but there are no novel eternal objects.
- (vii) That an eternal object can be described only in terms of its potentiality for `ingression' into the becoming of actual entities. The term `ingression' refers to the particular mode in which the potentiality of an eternal object is realized in a particular actual entity.
- (xi) Prehensions whose data involve actual entities are termed `physical prehensions'; and prehensions of eternal objects are termed 'conceptual prehensions'.

Eternal Objects are Atemporal

- Any entity whose conceptual recognition does not involve a necessary reference to any definite actual entities of the temporal world is called an 'eternal object'.
- Eternal objects are therefore time-invariant.
- (xv) That a proposition is the unity of certain actual entities in their potentiality for forming a nexus, with its potential relatedness partially defined by certain eternal objects which have the unity of one complex eternal object.

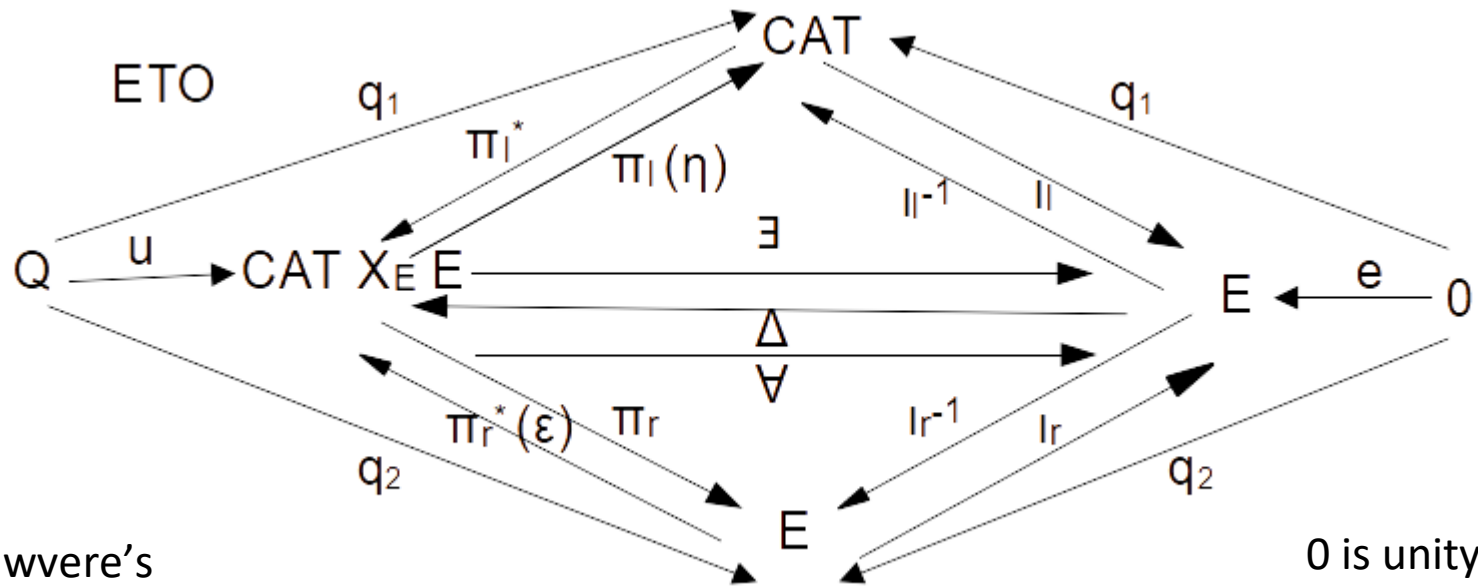
Examples of Eternal Objects

- Eternal objects include sensory qualities, like colours and tactile sensations; conceptual abstractions like shapes; numbers; moral qualities; physical fundamentals; feelings like an emotion, aversion, pleasure or pain; qualia
- Giving conceptual prehension in form of feelings

The Eternal Object Category as a locally Cartesian closed category ETO: Adjointness in the Functors between the Product $CAT \times_E E$ and the Category E

Δ is a constant functor

Actually a topos



Lawvere's hyperdoctrine

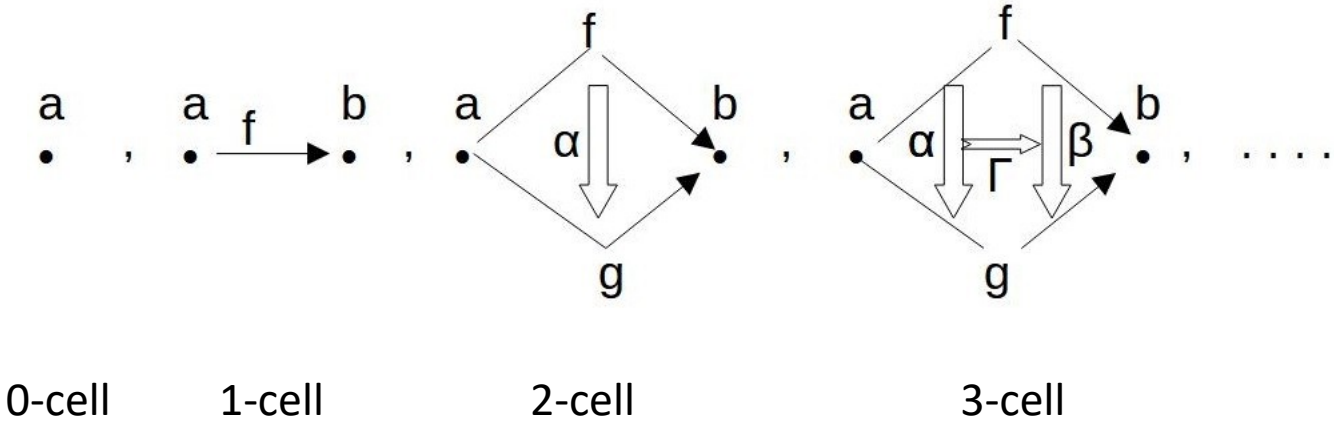
0 is unity of the one complex eternal object of Whitehead

Contrasts

Categories of Categories

- The eighth category includes an indefinite progression of categories, as we proceed from 'contrasts' to 'contrasts of contrasts' and on indefinitely to higher grades of contrasts.
- Contrast has a similar meaning to compare, a frequent description of functors and natural transformations in category theory.

n-cells in Higher-order Category Theory from Leinster (2004)



Propositions

Propositions and Logic

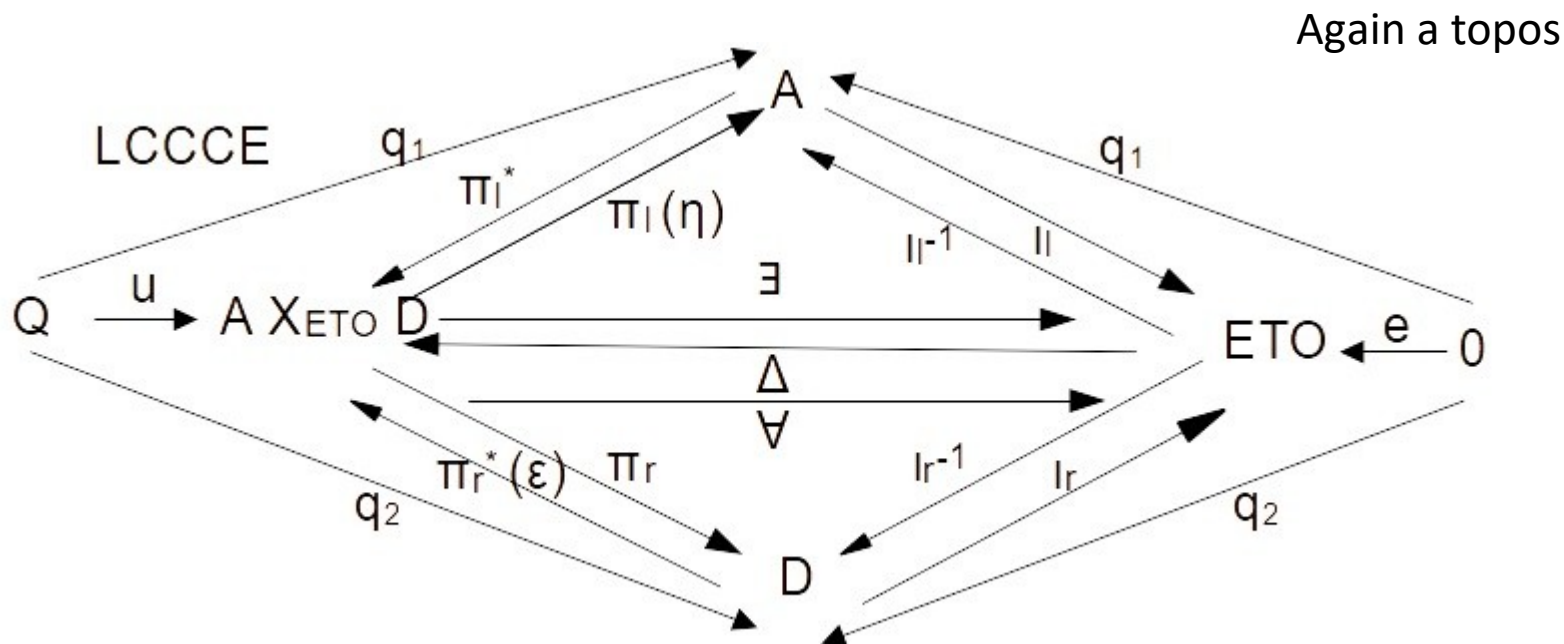
- Propositions imply logic
- Whitehead's logic is not like that encountered in set theory with subject, target and predicate
- He does though use the word predicate:
 - That a proposition is the unity of certain actual entities in their potentiality for forming a nexus, with its potential relatedness partially defined by certain eternal objects which have the unity of one complex eternal object. The actual entities involved are termed the 'logical subjects', the complex eternal object is the 'predicate'.

Propositions are Combinations of Features

- Some similarities to category theory.
- All the earlier constructions can be amalgamated by appropriate substitutions building complex constructions
- Logic is then provided by the Lawvere hyperdoctrine 'logic gate' through adjointness and the Heyting algebra

Conceptual Prehension with Concrecence

The locally Cartesian closed category LCCCE: Adjointness in the Functors between the Product $A \times_{\text{ETO}} D$ and the category ETO with 0 as the unity complex eternal object



Logic is in the arrows as 'logical subjects' and ETO as the 'predicate', governed by Lawvere's hyperdoctrine with adjointness $\exists \dashv \Delta \dashv \forall$. With ETO as eternal object, this is conceptual prehension

One Category of the Ultimate

Category of the Ultimate PR p.21

“ 'Creativity' 'many' 'one' are the ultimate notions involved in the meaning of the synonymous terms 'thing' 'being' 'entity'. These three notions complete the Category of the Ultimate and are presupposed in all the more special categories.

The term 'one' does not stand for 'the integral number *one*' which is a complex special notion. It stands for the general idea underlying alike the indefinite article '*a* or *an*' and the definite article '*the*' and the demonstratives '*this* or *that*' and the relatives '*which* or *what* or *how*.' It stands for the singularity of an entity. The term 'many' presupposes the term 'one' and the term 'one' presupposes the term 'many'. The term 'many' conveys the notion of 'disjunctive diversity'; this notion is an essential element in the concept of 'being'. There are many 'beings' in disjunctive diversity.”

Category of the Ultimate PR p.21 (continued)

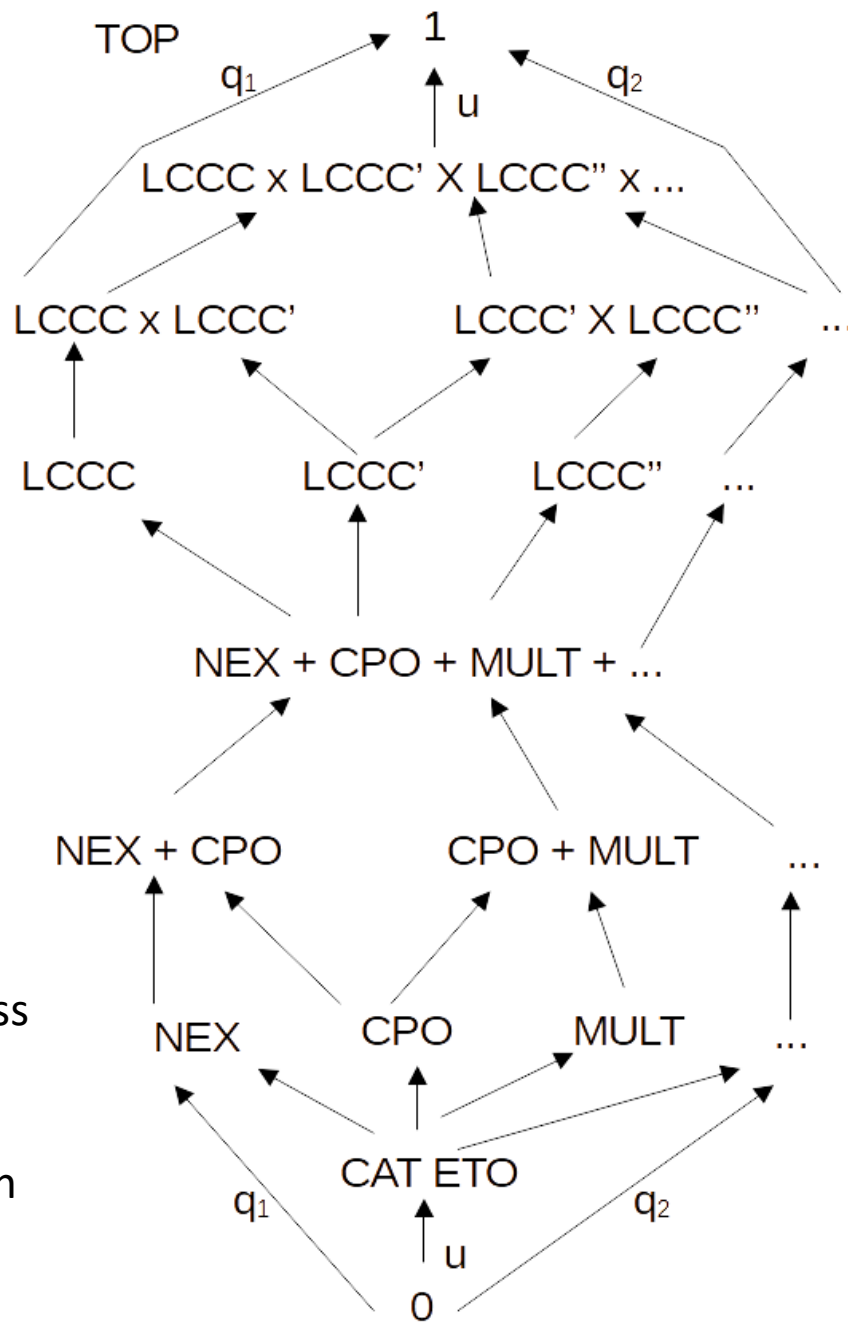
“ 'Creativity' is the universal of universals characterizing ultimate matter of fact. It is that ultimate principle by which the many, which are the universe disjunctively, become the one actual occasion, which is the universe conjunctively. It lies in the nature of things that the many enter into complex unity.”

Category of the Ultimate PR p.21 (continued)

“ The ultimate metaphysical principle is the advance from disjunction to conjunction, creating a novel entity other than the entities given in disjunction. The novel entity is at once the togetherness of the 'many' which it finds, and also it is one among the disjunctive 'many' which it leaves; it is a novel entity, disjunctively among the many entities which it synthesizes. The many become one, and are increased by one. In their natures, entities are disjunctively 'many' in process of passage into conjunctive unity. This Category of the Ultimate replaces Aristotle's category of 'primary substance’ “.

- Tension between X (conjunction) and + (disjunction) featured strongly in our last ANPA paper on music
- Aristotle's category of 'primary substance' is extensional
 - His secondary substance is intensional
 - Intension is an inherent part of each category through Dolittle diagrams (see our last ANPA paper on music)

The Topos
 Category TOP:
 Tension between
times (prehension)
 and *plus* (nexus)



Terminal object
 Limit cone
 One (singular)

Conjunctions
 Many

Prehension
 (pullback) with
 concrescence

Disjunctions
 Many

Nexus
 Multiplicities
 Prehension
 (pushout)

Actual Entities
 Eternal objects
 Many

Initial object
 Colimit cone
 One (singular)

Many more
 arrows exist across
 multiple levels.
 These are
 minimal collection

Categories of Explanation

Categoreal Obligations

The Categories of Explanation PR p.22

“ There are twenty-seven Categories of Explanation:

(i) That the actual world is a process, and that the process is the becoming of actual entities. Thus actual entities are creatures; they are also termed 'actual occasions' “

- Becoming is the creative process
- Occasions are Now (in time)

The Categories of Explanation PR p.22

“ (iv) That the potentiality for being an element in a real concrescence* of many entities into one actuality is the one general metaphysical character attaching to all entities, actual and non-actual; and that every item in its universe is involved in each concrescence. In other words, it belongs to the nature of a 'being' that it is a potential for every 'becoming'. This is the 'principle of relativity'. “

- That being is a potential for becoming

The Categories of Explanation PR pp.23-24

“ (xii) That there are two species of prehensions: (a) 'positive prehensions' which are termed 'feelings,' and (b) 'negative prehensions' which are said to 'eliminate from feeling.' Negative prehensions also have subjective forms. A negative prehension holds its datum as inoperative in the progressive concrescence of prehensions constituting the unity of the subject,

(xiii) That there are many species of subjective forms, such as emotions, valuations, purposes, aversions, aversions, consciousness, etc. “

- That positive prehensions include feelings, while negative prehensions do not include feelings.
- That subjective forms (of relationships) include emotions

The Categories of Explanation PR pp.25-26

“ (xxv) The final phase in the process of concrescence, constituting an actual entity, is one complex, fully determinate feeling. This final phase is termed the 'satisfaction'. It is fully determinate (a) as to its genesis, (b) as to its objective character for the transcendent creativity, and (c) as to its prehension—positive or negative—of every item in its universe.”

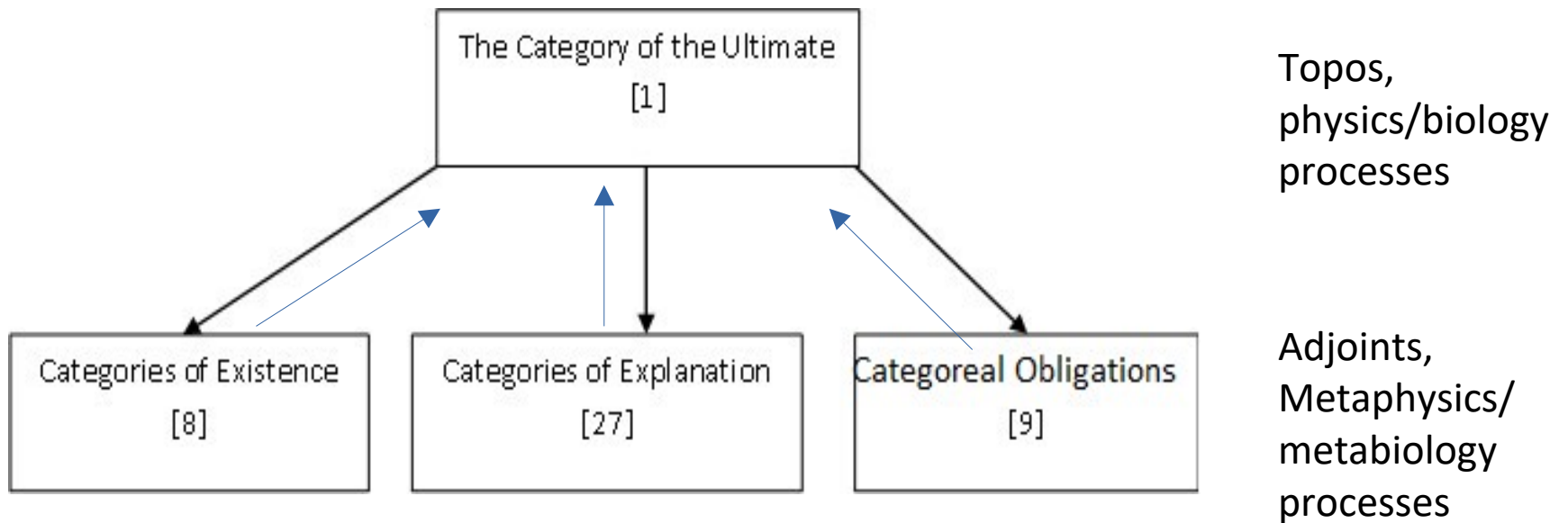
- That concrescence is a process in which prehensions are integrated into a fully determinate feeling or satisfaction.
 - Satisfaction and feelings are the closure of the prehension (and of the adjointness)

There are nine Categorical Obligations PR pp.26-27

- “ (i) The Category of Subjective Unity ...
- (ii) The Category of Objective Identity ...
- (iii) The Category of Objective Diversity ...
- (iv) The Category of Conceptual Valuation ...
- (v) The Category of Conceptual Reversion ...
- (vi) The Category of Transmutation ...
- (vii) The Category of Subjective Harmony ...
- (viii) The Category of Subjective Intensity ...
- (ix) The Category of Freedom and Determination ... “

- Concerned with maintenance of integrity

Overall Scheme Revisited



Numbers are count of categories of each type

Not a hierarchy but an adjointness

Results 1

- The Category of the Ultimate is a topos, handling the process ‘becoming’
 - In a single-substance philosophy
 - Mind and matter are treated as one
 - The match between Whitehead’s language and topos theory is very close
- The other special categories (Existence, Explanation, Obligations) expand and control ‘becoming’:
 - Can be viewed as metaphysics/metabiology

Results 2

- Whitehead makes extensive use of some terms of his own, which we have related to category theory:
 - ‘Prehension, relatedness’ – Cartesian/Cocartesian closed category
 - ‘Subjective Forms’ – pullback, pushout
 - ‘Nexus, togetherness’ – Cocartesian closed
 - ‘Concrescence’ – adjointness (Locally Cartesian closed category)
 - ‘Obligations’ – underlying functor (in adjointness)
 - ‘Satisfaction’ – adjoint conditions realised
 - ‘Category of the Ultimate’ – topos (Lawvere’s hyperdoctrine)

Results 3

- Categorification
 - Simple-minded translation 1:1 of concepts into category theory
- Have tried to avoid, looking at main thrust of Whitehead's work
 - Not sensible to translate someone's ideas expressed in great detail in words into my formalism
 - Not possible to understand all of Whitehead's text as he writes in an introspective way
- Still looking at later sections, with more explanation, and Whitehead's Harvard Lectures

Consequences

- Whitehead's *Process and Reality* provides a philosophical basis for category theory
- This is an alternative to the classical pure mathematical/set-based basis
- *Process and Reality* gives a much richer realism than set theory, facilitating
 - process as a single substance
 - a much richer type system, including complex metadata in the three additional special categories
 - a move towards biology
- A pure mathematician would say 'What are the types?'